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INTEGRATION OF SUBDIVISION METHOD INTO BOUNDARY ELEMENT ANALYSIS

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Subdivision surface modeling, which is based on polygon mesh modeling, can generate a whole smooth geometry without limiting to the topology and connectivity. Meanwhile, the boundary element analysis (BEA), which is based on the boundary integral equation, requires only boundary discretization of the body in question. Thus, performing BEA directly on the subdivision surface models may be a promising way to realize the seamless integration. This work presents a unified framework for the BEA and CAD modeling based on the subdivision surface. Numerical examples for 3D potential problems have demonstrated that the implementation is successful.

Keywords: Subdivision surface; loop subdivision; boundary element analysis.

1. Introduction

The traditional CAE analysis and CAD modeling are usually independent of each other. Models in CAE analysis, which is based on several numerical engineering methods including the finite difference method, the finite element method and the boundary element method, are discrete mesh models. Models in CAD modeling are continuous parametric models. Hence, the primary work of the CAE analysis is to convert the continuous model into a discrete model. During the conversion process, geometric errors are inevitably introduced, as the discrete model is actually an approximation of the continuous model. Meanwhile, the geometric error depends on the size and order of the elements that are used in the discrete model. To get better approximation, finer and a larger number of elements have to be used. This process is usually very time-consuming. Moreover, to apply the CAE results to design, it is often desirable to convert the discrete model back into the continuous

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model. This counter-conversion is even more difficult and, in most cases, almost impossible. Therefore, the integration of the CAD modeling and the CAE analysis has attracted intensive interests from both academic and engineering societies.

Many approaches have been proposed to link CAE and CAD seamlessly. Fehmi *et al.* [2002] combined the subdivision method and the finite element method to solve a shell problem. Hughes *et al.* [2005] proposed an isogeometry analysis that is based on NURBS. Cirak and Ortiz [2001] used surface elements for shell deformation problems. Zhang *et al.* [2009] proposed a boundary face method (BFM) that is based on the B-rep data structure that is widely applied in most CAD solid modeling packages. In the BFM, the geometric quantities such as the coordinates, outward normals, and Jacobians at Gaussian integral points are calculated directly from the parametric surface, and the integration and physical variable approximation are performed in the parametric spaces of the surfaces.

The subdivision surface is an alternative solid modeling method to parametric modeling. It generates a sequence of recursively refined meshes starting from an initial coarse control mesh. Subdivision surface can be considered as a generalization of spline surfaces. However, in contrast to spline surfaces, the subdivision surface modeling can represent structures of arbitrary topology without seaming and patching operation, namely, the subdivision surface is restricted neither by topological nor by geometric constraints as spline surfaces are [Zorin *et al.* (2000)]. Thus, it has been widely applied in geometric modeling, shape design, and surface reconstruction.

Since the subdivision surface and the boundary element analysis (BEA) are both performed based on the surface mesh, it is natural and convenient to perform BEA directly on the model generated by subdivision method. Wang [2009] proposed the idea of integration of CAD and BEA through subdivision surface. Unfortunately, the author did not provide any numerical examples.

This paper presents a new framework based on the subdivision surface to implement BEA, in which both BEA and CAD models are represented identically with the same subdivision model, realizing the integration of subdivision surface and BEA. This paper is organized as follows: in Sec. 2, the subdivision surface and the half-edge data structure used for representing and manipulating subdivision meshes are introduced. Section 3 details the BEA based on subdivision surface. The BEA and the steps in the implementation of BEA based on subdivision method are described in Sec. 4. Some numerical examples are given in Sec. 5. This paper ends with conclusions in Sec. 6.

2. Subdivision Surface and the Half-Edge Data Structure

2.1. *Subdivision surface*

Subdivision surface is a surface modeling technology that is based on polygon mesh. It can represent structures of arbitrary topology. Furthermore, the solid represented by subdivision surfaces is whole smooth. Due to its powerful modeling capabilities, it is widely applied in various areas including shape design and surface reconstruction.

The Loop subdivision scheme that is based on triangular meshes was proposed first by Charles Loop [1987]. It is a dual approximation subdivision. As a generalization of the Box Spline, the Loop subdivision scheme generates a C^2 continuous limit surface. Additionally, the loop subdivision scheme inherited the flexibility of representation from mesh modeling and the capability of representing arbitrary topology and geometry from the subdivision surface modeling. Hence, the loop subdivision scheme is studied in this paper, and its subdivision meshes are directly used for BEA.

2.2. *The half-edge data structure*

How to manage the subdivision mesh and to transfer the mesh information to BEA effectively are important issues, as the subdivision process will generate a large quantity of meshes. Here, we adopt a half-edge data structure and its function library that is called the OpenMesh. The OpenMesh library [Botsch *et al.* (2002)] consists of a number of functions that are efficient and versatile implementations of the half-edge data structure. Moreover, it is a generic and efficient data structure for representing and manipulating polygonal meshes.

The half-edge data structure, which has two half-edges opposite in direction by splitting each edge, is employed in this paper to store the subdivision meshes. The connectivity relations in the half-edge data structure are shown in Fig. 1.

As shown in Fig. 1, each vertex is related to an outgoing half-edge, i.e. the half-edge from the vertex (1); each face points to an incident half-edge (2); in addition, each half-edge provides several pointers, which point to the corresponding vertexes (3), the face it belongs to (4), the next half-edge inside the face (5), the opposite half-edge (6), and the previous in the face (7), respectively.

According to the connectivity relation between the mesh items, we are able to enumerate all its vertices, half-edges, and adjacent faces. An example of searching a vertex is illustrated in Fig. 2. We start from a mesh vertex. Then, we can search

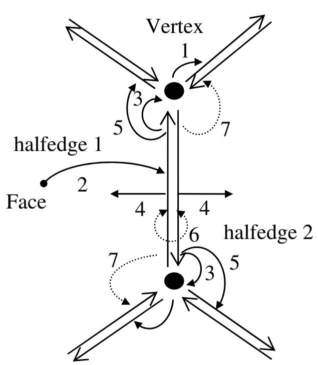


Fig. 1. The half-edge data structure.

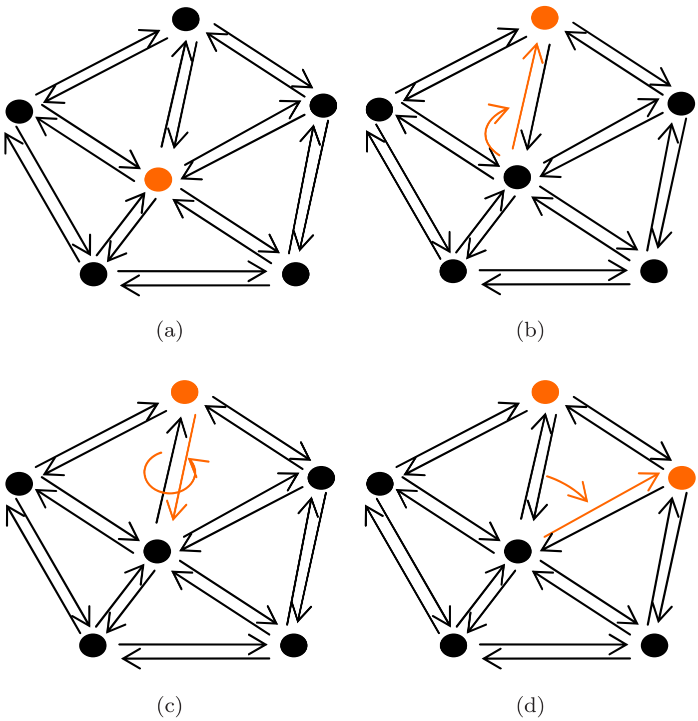


Fig. 2. Mesh items traversing: (a) Start at a vertex, (b) outgoing half-edge, (c) the opposite half-edge, and (d) vertex the next-half edge pointing to.

for the vertex to which the outgoing half-edge of the vertex points. The opposite half-edge can be found in the third step. Finally, we can get the neighbor vertex to which the opposite half-edge points. We can repeat steps (b)–(d), until all the neighbor vertices are found out. The adjacent faces can also be found analogously.

3. BEA Based on Subdivision Surface

The BEA [Brebbia *et al.* (1988); Long (2002)] is based on boundary integral equation of boundary value problems and requires discretization of only the boundary. Thus, it simplifies the analysis to a large extent by solving a small system of algebraic equations [Kythe (1995)], and reduces the dimensionality of a problem by one.

In traditional BEA, the discretization of the boundary of geometry models, namely surface meshes generation, is inevitable. The quality of the meshes mainly depends on the user's experience and intuition, which is the main source of the analysis error. The accuracy and convergence of the BEA also depend on the quality of the meshes. Thus, an automatic and adaptive mesh generation for BEA is important.

Subdivision surface uses the surface mesh for representation of models, and BEA only need surface mesh generation of models. Therefore, subdivision surface itself can provide a geometric model for the BEA. Meanwhile, the adaptive subdivision scheme is able to provide an automatic and adaptive BEA mesh generation scheme. Thus, we use the subdivision meshes for BEA, coupling the CAD, and BEA through subdivision surface.

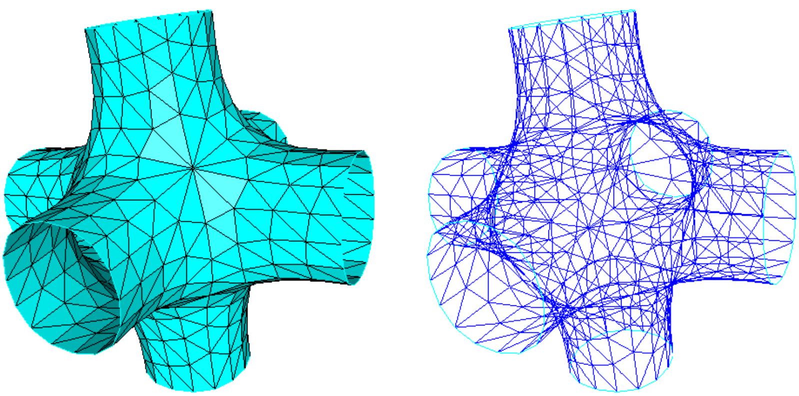
3.1. *Creating initial control mesh*

The first step of BEA based on subdivision surface is creating the initial control mesh. The creation of the initial control mesh depends on the body's features, which can be classified into form features and transitional features. The former refers to basic planes, cylinder surfaces *et al.* The latter refers to features concatenating these form features. Designers can use some interactive design packages for subdivision surface modeling, such as Maya and 3DMAX, which allow designers to input and edit the initial control mesh in terms of their design ideas.

3.2. *BEA mesh generation*

Traditional BEA mesh generation depends mainly on analysts. Different users with different mesh generations may get different analysis results. For a complicate geometry, the traditional BEA is time-consuming and there is no guarantee that the result is sufficiently accurate. Thus, there is a need to develop an automatic and adaptive BEA mesh generation scheme.

In our method, the subdivision mesh not only provides a representation for geometry models, but also can be used as BEA mesh. As shown in Fig. 3(a), subdivision meshes are used for representing geometry models. In Fig. 3(b), the meshes are used for BEA. They are the same subdivision meshes. Adaptive subdivision scheme, determining the subdivision times in terms of local flatness information, can generate accurate subdivision meshes in accordance with geometry models and



(a) (b)

Fig. 3. Subdivision mesh and BEA mesh.

fulfill adaptive BEA meshes. Moreover, subdivision surface can generate meshes at different levels (from coarse to fine) to meet different requirements in accuracy of geometry models and BEA.

4. Boundary Element Analysis

4.1. *BEA description for the 3D potential problem*

We take the 3D potential problem as an example to describe the BEA process. The potential problem in three dimensions [Qin *et al.* (2010)] governed by Laplace's equation with boundary conditions is written as

$$\begin{aligned} u_{,ii} &= 0, & \forall x \in \Omega \\ u &= \bar{u}, & \forall x \in \Gamma_u \\ u_{,i}\, n_i \equiv q &= \bar{q}, & \forall x \in \Gamma_q, \end{aligned} \tag{1}$$

where the domain Ω is enclosed by $\Gamma = \Gamma_u + \Gamma_q$, $\bar{u}$ and $\bar{q}$ are the given potential and normal flux, respectively, on the essential boundary Γ_u and on the flux boundary Γ_q, and n is the outward normal direction to the boundary Γ, with components n_i,$i = 1, 2, 3$.

The problem can be recast into an integral equation on the boundary. The well-known self-regular BIE for potential problems in 3D is

$$0 = \int_\Gamma (u(\mathbf{s}) - u(\mathbf{y}))q^s(\mathbf{s}, \mathbf{y})\mathrm{d}\Gamma - \int_\Gamma q(\mathbf{s})u^s(\mathbf{s}, \mathbf{y})\mathrm{d}\Gamma, \tag{2}$$

where $q = \partial u/\partial n$, $\mathbf{y}$ is the source point, and $\mathbf{s}$ is the field point on the boundary. $u^s(s, y)$ and $q^s(s, y)$ are the fundamental solutions. For 3D potential problems,

$$u^s(\mathbf{s}, \mathbf{y}) = \frac{1}{4\pi}\frac{1}{r(\mathbf{s}, \mathbf{y})}, \tag{3}$$

$$q^s(\mathbf{s}, \mathbf{y}) = \frac{\partial u^s(\mathbf{s}, \mathbf{y})}{\partial n}, \tag{4}$$

with $r(s, y)$ being the Euclidean distance between the source and field points.

4.2. *Steps in the implementation of BEA based on subdivision method*

Subdivision surface generates a series of meshes converging into a whole smooth limit surface. The geometry data can be calculated by the subdivision scheme used. We use the subdivision mesh directly for BEA. This is the basic idea of the BEA based on subdivision surface. The following steps are summarized for the implementation:

(1) Creating the initial control mesh in terms of the geometry model.

(2) Creating fine meshes through an iterative refinement process using the Loop subdivision, until an ideal geometry model with sufficient accuracy of the part is obtained.
(3) Using the subdivision mesh for BEA directly.
(4) Capable of shape optimizing for the subdivision model in accordance with the BEA result.

5. Numerical Examples for 3D Potential Problems

In this section, we solve the Laplace equation

$$\nabla^2 u = 0, \tag{5}$$

on two bodies of different geometries. In order to assess the accuracy of the method, we have used the following cubic analytical field:

$$u = x^3 + y^3 + z^3 - 3yx^2 - 3xz^2 - 3zy^2. \tag{6}$$

For the purpose of error estimation and convergence study, a "global" L_2 norm error, normalized by $|v|_{\max}$ is defined as

$$e = \frac{1}{|v|_{\max}} \sqrt{\frac{1}{N} \sum_{i=1}^{N} (v_i^{(e)} - v_i^{(n)})^2}, \tag{7}$$

where $|v|_{\max}$ is the maximum value of u and q over N sample point and the superscripts (e) and (n) refer to the exact and numerical solutions, respectively.

5.1. *A trimmed cubic*

Figure 4 shows a trimmed cubic model, with the modeling process from the initial control mesh to the subdivision model of third level. The relative errors for normal fluxes at all boundary nodes are listed in Table 1. Due to the memory limitation of the computer resource, the last model in Fig. 4 has not been analyzed for its too large number of nodes.

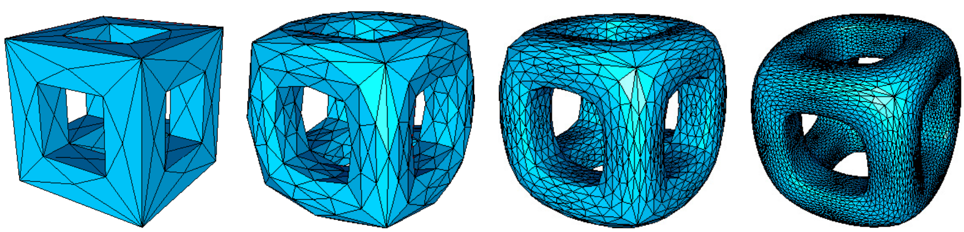


Fig. 4. Subdivision models of a trimmed cubic.

Table 1. BEA analysis result for a trimmed cubic.

Subdivision times	0	1	2
Vertices	88	376	1,528
Elements	192	768	3,072
Error (%)	33.06	4.382	1.597

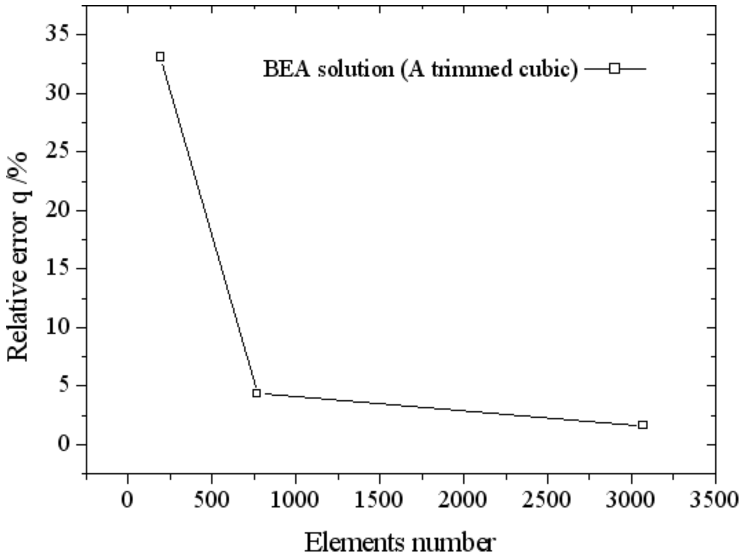


Fig. 5. Relative errors of nodal q and convergence rates.

From Table 1, it is seen that acceptable results have been obtained using 376 vertices. The convergence rate is shown in Fig. 5. Although the results obtained by the initial mesh are not acceptable, the numerical solution converges to the exact solution quickly. For clarity, the normal flux on the boundary of the body is shown in Fig. 6.

5.2. *A mushroom*

Figure 7 shows a mushroom, with the modeling process from the initial control mesh to the subdivision model of third level. We have performed BEA for each subdivision mesh. Results are presented in Table 2.

For this complicated geometry, acceptable results have been obtained using 242 vertices and very accurate results obtained using 3,842 vertices. The relative error reduces dramatically as increasing numbers of vertices are used. The convergence rate is shown in Fig. 8. The distribution of normal flux is illustrated in Fig. 9. Again, the numerical solutions converge to the exact solution quickly. From Table 2, we can see that the mesh quantities and the analysis accuracy increase four times by once

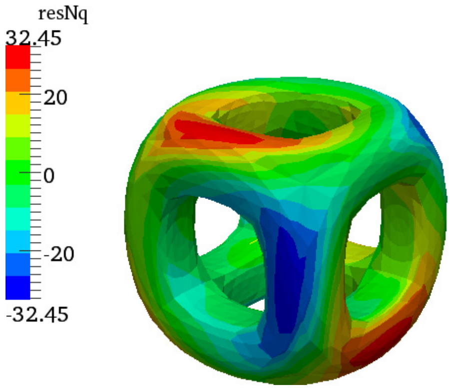


Fig. 6. Contour for normal flux q.

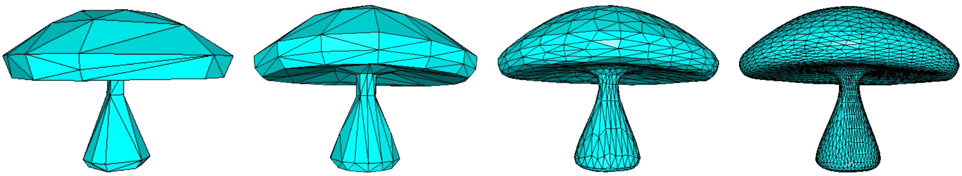


Fig. 7. Subdivision models of a mushroom.

Table 2. BEA analysis results for a mushroom.

Subdivision times	0	1	2	3
Vertices	62	242	962	3,842
Elements	120	480	1,920	7,680
Error (%)	13.57	4.844	0.73	0.24

subdivision. Most importantly, our method can perform BEA at every subdivision level automatically. This feature is particularly useful for adaptive analysis.

6. Conclusions

The subdivision surface and BEA analysis are integrated into a unified framework in this paper. The subdivision surface can be applied to represent structures of arbitrary topology without seaming and patching operations, thus the combined method is able to analyze complicated structures automatically.

Two numerical examples are presented. High convergence rates have been achieved. The results of normal fluxes for the two examples are accurate, according to their relative errors. However, the examples presented in this paper are just results at the primary stage of our research.

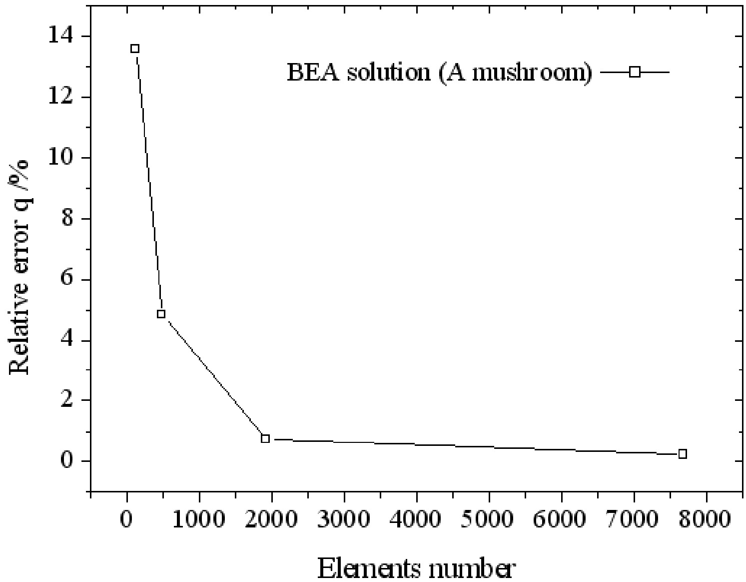


Fig. 8. Relative errors of nodal q and convergence rates.

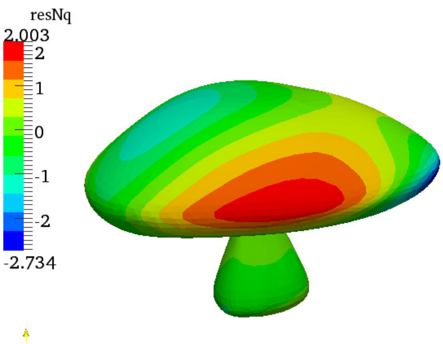


Fig. 9. Normal flux q contour.

The most exciting feature of our method, perhaps, is that it unifies the geometric modeling and CAE into a unique framework and thus has potential to offer very promising applications in practical engineering. The only drawback is that the quality of the subdivision mesh is not good enough to get acceptable analysis accuracy in some cases. How to avoid this pitfall is a planned investigation in our future research. Nevertheless, the advantages are so attractive that this method deserves consideration.

Combining our method with the fast multipole method [Zhang *et al.* (2010)] to implement BEA for large-scale subdivision mesh is also a main topic in our future work.

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References

Botsch, M., Steinberg, S., Bischoff, S. and Kobbelt, L. [2002] OpenMesh — a generic and efficient polygon mesh data structure, *In OpenSG Symposium* 2002.

Brebbia, C. A. *et al.* [1988] LONG shu-yao, *et al.* translated, *The Theory and Engineering Application of the Boundary Element Method*, National Defence Industry Press, Changsha.

Cirak, F. and Ortiz, M. [2001] Fully C1-conforming subdivision elements for finite deformation thin shell analysis, *Int. J. Numer. Meth. Eng.* **51**, 813–833.

Fehmi, C., Michael, J. S., Erik, K. A., Ortiz, M. and Schroder, P. [2002]. Integrated modeling, finite-element analysis, and engineering design for thin-shell structures using subdivision, *Comput. Aided Des.* **34**, 137–148.

Hughes, T. J. R., Cottrell, J. A. and Bazilevs, Y. [2005] Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, *Comput. Meth. Appl. Mech. Eng.* **194**, 4135–4195.

Kythe, P. K. [1995] *An Introduction to Boundary Element Methods*, CRC Press, Inc., Boca Raton.

Loop, C. T. [1987] Smooth subdivision surfaces based on triangles, *M.S. Thesis*, Department of Mathematics, University of Utah, August 1987.

Long, S. Y. [2002] *The Conspectus of the Boundary Element Method*, China Science and Culture Press (in Chinese).

Qin, X. Y., Zhang, J. M., Li, G. Y. and Sheng, X. M. [2010] An element implementation of the boundary face method for 3D potential problems, *Eng. Anal. Bound. Elem.* **34**, 934–943.

Wang, L. [2009] Integration of CAD and boundary element analysis through subdivision methods, *Compu. Ind. Eng.* **57**, 691–698.

Zhang, J. M., Qin, X. Y., Han, X. and Li, G. Y. [2009] A boundary face method for potential problems in three dimensions, *Int. J. Numer. Meth. Eng.* **80**, 320–337.

Zhang, J. M., Zhuang, Ch., Qin, X. Y., Li, G. Y. and Sheng, X. M. [2010] FMM-accelerated hybrid boundary node method for multi-domain problems, *Eng. Anal. Bound. Elem.* **34**, 433–439.

Zorin, D., Schröder, P., DeRose, T., Kobbelt, L., Levin, A. and Sweldens, W. [2000] Subdivision for modeling and animation, *In Course Notes of ACM SIGGRAPH* 00.